1st SolarNet school: Polarimetry basis

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Overview

1. Part I: 10:00-10:45
   - Polarized light, Stokes vector $\mathbf{I} = (I, Q, U, V)$
   - radiative transfer equation for polarized light (RTE)
2. Part II: 11:00-11:45
   - Absorption and dispersion
   - Classical and Quantum-mechanical Zeeman effect
3. Part III: 13:30-14:15
   - Simple analytical solutions
   - Numerical solvers

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Stokes vector:

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Take an EM plane-wave:

$$\mathbf{E}(\mathbf{r},t) = \underbrace{\varepsilon_x e^{i\delta_x} e^{i\mathbf{kr}} e^{-iwt}}_{E_x \in \mathbb{C}} \mathbf{e}_x + \underbrace{\varepsilon_y e^{i\delta_y} e^{i\mathbf{kr}} e^{-iwt}}_{E_y \in \mathbb{C}} \mathbf{e}_y \quad \varepsilon_x, \varepsilon_y \in \mathbb{R} \tag{1}$$

Important for later is the dispersion relation:

$$k = f(w)$$

Stokes vector $\mathbf{I}$ is defined as:

$$\mathbf{I} = (I, Q, U, V)^\dagger$$

where:

$$\begin{array}{ll} I = (E_x E_x^* + E_y E_y^*) & Q = (E_x E_x^* - E_y E_y^*) \\ U = (E_x E_y^* + E_y E_x^*) & V = (E_x E_y^* - E_y E_x^*) \end{array} \tag{2}$$

Defining $\delta = \delta_x - \delta_y$ (phase lag between components):

$$\mathbf{I} \stackrel{\text{def}}{:=} \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} \stackrel{\text{def}}{:=} m \begin{pmatrix} \varepsilon_x^2 + \varepsilon_y^2 \\ \varepsilon_x^2 - \varepsilon_y^2 \\ 2\varepsilon_x\varepsilon_y \cos\delta \\ 2\varepsilon_x\varepsilon_y \sin\delta \end{pmatrix} \tag{3}$$

Polarization of an EM wave:

Start from Eq.(1) and define:

$$\alpha = \mathbf{Kr} - wt + \delta_x$$
$$\delta = \delta_x - \delta_y$$

Take real parts of x- and y-components:

$$\mathrm{Re}[E_x] = E_{rx} = \varepsilon_x \cos\alpha$$
$$\mathrm{Re}[E_y] = E_{ry} = \varepsilon_y(\cos\alpha\cos\delta + \sin\alpha\sin\delta)$$

Prove that:

$$\frac{E_{rx}^2}{\varepsilon_x^2} + \frac{E_{ry}^2}{\varepsilon_y^2} - 2\frac{E_{rx}E_{rx}}{\varepsilon_x\varepsilon_x}\cos\delta = \sin^2\delta \quad (4)$$

What does Eq.(4) remind you of ?

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Linear polarization

Make $|\delta| = 0, \pi$ in Eq.(4):

$$E_{rx}^2 - 2\left(\frac{\varepsilon_x}{\varepsilon_y}\right) E_{ry} E_{rx} + \left(\frac{\varepsilon_x}{\varepsilon_y}\right)^2 E_{ry}^2 = 0$$

This is a second degree equation in either E_{rx} or E_{yr}. Solving i.e. for E_{rx}:

$$E_{rx} = \pm\frac{\varepsilon_x}{\varepsilon_y} E_{ry}$$

E draws in time a straightline in the XY plane: **Linear polarization**.
Eq.(3) becomes:

$$\mathbf{I} = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} \varepsilon_x^2 + \varepsilon_y^2 \\ \varepsilon_x^2 - \varepsilon_y^2 \\ 2\varepsilon_x\varepsilon_y \cos\delta \\ 0 \end{pmatrix}$$

$$V = 0,\ Q, U \neq 0$$

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Circular polarization

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Make $|\delta| = \pi/2, 3\pi/2$ and $\varepsilon_x = \varepsilon_y = \varepsilon_0$ in Eq.(4):

$$E_{rx}^2 + E_{ry}^2 = \varepsilon_0^2$$

E draws in time a circle in the XY plane: **Circular polarization**.

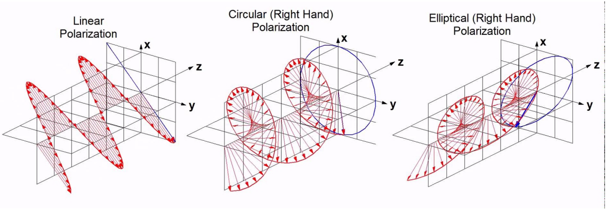
Eq.(3) becomes:

$$\mathbf{I} = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} \varepsilon_x^2 + \varepsilon_y^2 \\ 0 \\ 0 \\ \pm 2\varepsilon_x\varepsilon_y \end{pmatrix}$$

$$Q, U = 0, \; V \neq 0$$

Elliptical polarization

All other values of δ in Eq.(4), **E** draws an ellipse in time in the XY plane. $Q, U, V \neq 0$



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How is polarization in EM waves induced ?

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Polarization is induced by introducing phase lags between E_x & E_y

Phase lags are introduced by having different phase speed along x and y:

$$v_{ph,x} = c/n_x \neq v_{ph,y} = c/n_y$$
$$n_x \neq n_y$$

n_x and n_y are the refraction indexes along the x and y axis.

- In non-conducting crystals this occurs naturally as they are intrinsically anisotropic.
- In conducting media (such as the plasma in solar/stellar atmosphere) the presence of an external magnetic field $\mathbf{B}_{\rm ext}$ introduces anisotropy even if the medium was originally isotropic.

The RTE for polarized light

To fully characterized the properties of light and its variation along z we need the derivative of the Stokes vector with z (Eq.2):

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(E_x E_x^* + E_y E_y^*\right) = \frac{\mathrm{d}I}{\mathrm{d}z}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(E_x E_x^* - E_y E_y^*\right) = \frac{\mathrm{d}Q}{\mathrm{d}z}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(E_x E_y^* + E_y E_x^*\right) = \frac{\mathrm{d}U}{\mathrm{d}z}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(E_x E_y^* - E_y E_x^*\right) = \frac{\mathrm{d}V}{\mathrm{d}z}$$

Write the RTE as a conservation equation

$$\frac{\mathrm{d}\mathbf{I}}{\mathrm{d}z} = \text{sources} - \text{sinks}$$

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Sources: $\mathbf{j}$

$$\mathbf{j} \propto B(T,\nu) = \frac{n_u A_{ul} \Psi_\nu}{n_l B_{lu} \Phi_\nu - n_u B_{ul} \Psi_\nu} \tag{5}$$

Ψ_ν = emission profile

Φ_ν = absorption profile

B_{ul}, B_{lu}, A_{lu} = Einstein's coefficients

n_u, n_l = electron population of the energy level

$B(T,\nu)$ = Source function

Absorption/emission rates: Einstein's coefficients

$$B_{ul} = \frac{g_l}{g_u} B_{lu} \quad \frac{A_{ul}}{B_{ul}} = \frac{2h\nu^3}{c^2} \tag{6}$$

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Sources under Local Thermodynamic Equilibrium (LTE)

Under LTE:

- Complete redistribution:

$$\Phi_\nu = \Psi_\nu \quad (7)$$

- n_u & n_l given by Saha's equation:

$$\frac{n_u}{n_l} = \frac{g_u}{g_l} e^{-\frac{h\nu}{KT}} \quad (8)$$

- Substitute Eqs.(6,7,8) into Eq.(5)

$$B(T,\nu) = \frac{2h\nu^3}{c^2} \frac{1}{\exp \frac{h\nu}{KT} - 1} \quad (9)$$

- Sources: Planck's function

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Sources under Non-Local Thermodynamic Equilibrium (NLTE)

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Under Non-LTE:

- Partial redistribution: $\Phi_\nu \neq \Psi_\nu$
- n_u & n_l given by Statistical Equilibrium Equations
- See Jaime's lectures on Thursday & Friday

Sinks

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If there is no light, it cannot be absorbed:

$$\begin{aligned}\text{Sinks} &\propto \mathbf{I} \\ \text{Sinks} &= \hat{\mathcal{K}}\mathbf{I}\end{aligned}$$

$\hat{\mathcal{K}}$ is the propagation matrix:

$$\hat{\mathcal{K}} = \begin{pmatrix} \eta_I & \eta_Q & \eta_U & \eta_V \\ \eta_Q & \eta_I & \rho_V & -\rho_U \\ \eta_U & -\rho_V & \eta_I & \rho_Q \\ \eta_V & \rho_U & -\rho_Q & \eta_I \end{pmatrix}$$

Finally, the RTE:

$$\frac{\mathrm{d}\mathbf{I}}{\mathrm{d}z} = \text{sources} - \text{sinks} = \mathbf{j} - \hat{\mathcal{K}}\mathbf{I} \tag{10}$$

Elements of the propagation matrix

$$\hat{\mathcal{K}} = \begin{pmatrix} \eta_I & \eta_Q & \eta_U & \eta_V \\ \eta_Q & \eta_I & \rho_V & -\rho_U \\ \eta_U & -\rho_V & \eta_I & \rho_Q \\ \eta_V & \rho_U & -\rho_Q & \eta_I \end{pmatrix}$$

- η_I: pure absorption ; η_Q, η_U, η_V: dichroism $\Rightarrow$ different polarization states are absorbed differentely.
- η's are related to the imaginary part of the dispersion relation of an EM wave travelling through a medium: $k = f(w)$.

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Elements of the propagation matrix

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$$\hat{\mathcal{K}} = \begin{pmatrix} \eta_I & \eta_Q & \eta_U & \eta_V \\ \eta_Q & \eta_I & \rho_V & -\rho_U \\ \eta_U & -\rho_V & \eta_I & \rho_Q \\ \eta_V & \rho_U & -\rho_Q & \eta_I \end{pmatrix}$$

- ρ_V: Faraday pulsation transforms $Q, U \Rightarrow V$.
- ρ_Q, ρ_U: Faraday rotation transforms $Q \Longleftrightarrow U$.
- ρ's are related to the real part of the dispersion relation of an EM wave travelling through a medium: $k = f(w)$.

Elements of the propagation matrix

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Their derivation is very long so I just write them here:

$$\eta_I = \frac{1}{2}\left\{\chi_0 \sin^2\gamma + \frac{\chi_{+1}+\chi_{-1}}{2}(1+\cos^2\gamma)\right\}$$

$$\eta_Q = \frac{1}{2}\left\{\chi_0 - \frac{\chi_{+1}+\chi_{-1}}{2}\right\}\sin^2\gamma\cos 2\varphi \tag{11}$$

$$\eta_U = \frac{1}{2}\left\{\chi_0 - \frac{\chi_{+1}+\chi_{-1}}{2}\right\}\sin^2\gamma\sin 2\varphi$$

$$\eta_V = \frac{1}{2}(\chi_{+1}-\chi_{-1})\cos\gamma$$

and,

$$\rho_Q = \frac{1}{2}\left\{\widetilde{\chi}_0 - \frac{\widetilde{\chi}_{+1}+\widetilde{\chi}_{-1}}{2}\right\}\sin^2\gamma\cos 2\varphi$$

$$\rho_U = \frac{1}{2}\left\{\widetilde{\chi}_0 - \frac{\widetilde{\chi}_{+1}+\widetilde{\chi}_{-1}}{2}\right\}\sin^2\gamma\sin 2\varphi \tag{12}$$

$$\rho_V = \frac{1}{2}(\widetilde{\chi}_{+1}-\widetilde{\chi}_{-1})\cos\gamma$$

Origins of η & ρ

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- Dispersion relation of an EM plane wave: $k = f(w)$
- If the wave travels through vacuum: $k = w/c$, where c is the speed of light
- If it travels through a conducting medium there is a real k_r and imaginary k_i part in the dispersion relation:

$$k = k_r(\varepsilon) + \mathrm{i}k_i(\sigma)$$

- where ε is the dielectric constant that relates $\mathbf{D} = \varepsilon \mathbf{E}_{\rm ext} \Rightarrow \mathbf{D}$ is the displacement vector.
- where σ is the electrical conductivity that relates $\mathbf{j} = \sigma \mathbf{E}_{\rm ext} \Rightarrow \mathbf{j}$ is the electric current vector; $\mathbf{E}_{\rm ext}$ is an external electric field, not to be confused with the electric field of the passing EM wave.

Origins of η & ρ

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- Introducing the real k_r and imaginary k_i part in the dispersion relation into the EM plane wave solution Eq(1) we get something like:

$$e^{i\mathbf{kr}} \propto \overbrace{e^{ik_r(\varepsilon)}}^{\text{oscillation}} \underbrace{e^{-k_i(\sigma)}}_{\text{exp.decay}}$$

- The dielectric constant ε introduces phase lags between E_x and E_y in the passing EM wave $\Rightarrow$ changes the polarization state
- The conductivity σ decreases the amplitude of the electric field of the passing EM wave. This is done by means of Joule dissipation of electric currents $\mathbf{j}$.

Origins of η & ρ

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Back to:

$$\mathbf{D} = \hat{\varepsilon}\mathbf{E}_{\mathrm{ext}} \quad ; \ \mathbf{j} = \hat{\sigma}\mathbf{E}_{\mathrm{ext}}$$

- If the medium is isotropic $\hat{\sigma} = \sigma\hat{\mathbb{I}}$ and $\hat{\varepsilon} = \varepsilon\hat{\mathbb{I}}$ are scalars.
- If the medium is anisotropic $\hat{\sigma}$ and $\hat{\varepsilon}$ are matrices.
- And remember: even if medium is isotropic, an external magnetic field $\mathbf{B}_{\mathrm{ext}}$ will induce anisotropy (see later).

$\mathbf{B}_{\rm ext}$ orientation: γ & φ

$$\eta_I = \frac{1}{2}\left\{\chi_0 \sin^2\gamma + \frac{\chi_{+1}+\chi_{-1}}{2}(1+\cos^2\gamma)\right\}$$

$$\eta_Q = \frac{1}{2}\left\{\chi_0 - \frac{\chi_{+1}+\chi_{-1}}{2}\right\}\sin^2\gamma\cos 2\varphi$$

$$\eta_U = \frac{1}{2}\left\{\chi_0 - \frac{\chi_{+1}+\chi_{-1}}{2}\right\}\sin^2\gamma\sin 2\varphi$$

$$\eta_V = \frac{1}{2}(\chi_{+1}-\chi_{-1})\cos\gamma$$

and,

$$\rho_Q = \frac{1}{2}\left\{\widetilde{\chi}_0 - \frac{\widetilde{\chi}_{+1}+\widetilde{\chi}_{-1}}{2}\right\}\sin^2\gamma\cos 2\varphi$$

$$\rho_U = \frac{1}{2}\left\{\widetilde{\chi}_0 - \frac{\widetilde{\chi}_{+1}+\widetilde{\chi}_{-1}}{2}\right\}\sin^2\gamma\sin 2\varphi$$

$$\rho_V = \frac{1}{2}(\widetilde{\chi}_{+1}-\widetilde{\chi}_{-1})\cos\gamma$$

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$\mathbf{B}_{\text{ext}}$ orientation: γ & φ

Write the external magnetic field vector in spherical coordinates in the observer's referece frame:

$\mathbf{B}_{\text{ext}} = (B, \gamma, \varphi)$

- B: modulus of the external magnetic field.
- γ: inclination of the magnetic field with respect to the observer's reference frame.
- φ: azimuth of the magnetic field in the plane perpendicular to the observer's reference frame.

$\mathbf{B}_{\text{ext}}$ appears because it is the cause the medium is anisotropic. Let us see how.

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How does $\mathbf{B}_{\rm ext}$ induce anisotropy ?

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Electric field **E**, displacement vector **D** and Dipole moment **P**

$$\mathbf{D} = \mathbf{E}_{\rm ext} + 4\pi\mathbf{P}$$

P is the macroscopic dipole moment (charge$\times$distace) of the charges in the conducting medium induced by the external electric $\mathbf{E}_{\rm ext}$ and magnetic $\mathbf{B}_{\rm ext}$ fields:

$$\mathbf{P} = -eN\mathbf{r}$$

where **r** is the displacement of the charges in the conducting medium (from their equilibrium position) induced by the external fields:

$$\mathbf{D} = \mathbf{E}_{\rm ext} + 4\pi\mathbf{P} = \mathbf{E}_{\rm ext} - 4\pi Ne\mathbf{r} = \hat{\mathcal{E}}\mathbf{E}_{\rm ext} \quad (13)$$

To find $\hat{\mathcal{E}}$ all we need to do is find $\mathbf{r}(\mathbf{E}_{\rm ext})$:

- if $\mathbf{r}(\mathbf{E}_{\rm ext}) \parallel \mathbf{E}_{\rm ext}$ then $\hat{\mathcal{E}}$ is an scalar and the medium is isotropic
- otherwise $\hat{\mathcal{E}}$ is a matrix and the medium is anisotropic

Finding $\mathbf{r}(\mathbf{E}_{\rm ext})$: How does $\mathbf{B}_{\rm ext}$ induce anisotropy ?

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Classical equation of motion of an electron in an atom subject to an external EM field

$$m\ddot{\mathbf{r}} = -\underbrace{k\mathbf{r}}_{\text{restoring}} - \underbrace{e\mathbf{E}_{\rm ext}}_{\text{electrostatic}} - \underbrace{m\Gamma\dot{\mathbf{r}}}_{\text{damping}} - \underbrace{\frac{e}{c}\dot{\mathbf{r}} \times \mathbf{B}_{\rm ext}}_{\text{Lorentz}} \quad (14)$$

- We assume $\mathbf{E}_{\rm ext}$ and $\mathbf{B}_{\rm ext}$ do not depend on $\mathbf{r}$.
- We propose a solution in which the electron oscillates around its equilibrium position $\mathbf{r}_0$ (i.e. bounded energy level in the atom).
- The x-component (for instace) of this solution is:

$$\begin{aligned} r_x &= r_{x0}{\rm e}^{-iwt} \\ \dot{r}_x &= -iwr_x \\ \ddot{r}_x &= -w^2 r_x \end{aligned}$$

Finding $\mathbf{r}(\mathbf{E}_{\rm ext})$: How does $\mathbf{B}_{\rm ext}$ induce anisotropy ?

- Substitute this solution into the x-component of the equation of movement (remove ext for convenience):

$$-mw^2 r_x = -kr_x - eE_x - m\Gamma \dot{r}_x - \frac{e}{c}[\dot{r}_y B_z - \dot{r}_z B_y]$$
$$= -kr_x - eE_x + imw\Gamma r_x + i\frac{ew}{c}[r_y B_z - r_z B_y]$$

- If there was no magnetic induction field we could write:

$$\frac{E_x}{r_x} = \frac{1}{e}(mw^2 + imw\Gamma - k) = \frac{E_y}{r_y} = \frac{E_z}{r_z}$$

And therefore $\mathbf{E}_{\rm ext} \parallel \mathbf{r}$ (isotropy)

- Unfortunately, the presence of the external magnetic induction field $\mathbf{B}_{\rm ext}$ makes $\mathbf{E}_{\rm ext} \nparallel \mathbf{r}$ (anisotropy)

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Absorption & dispersion: χ & $\widetilde{\chi}$

$$\eta_I = \frac{1}{2}\left\{\chi_0 \sin^2\gamma + \frac{\chi_{+1}+\chi_{-1}}{2}(1+\cos^2\gamma)\right\}$$

$$\eta_Q = \frac{1}{2}\left\{\chi_0 - \frac{\chi_{+1}+\chi_{-1}}{2}\right\}\sin^2\gamma\cos 2\varphi$$

$$\eta_U = \frac{1}{2}\left\{\chi_0 - \frac{\chi_{+1}+\chi_{-1}}{2}\right\}\sin^2\gamma\sin 2\varphi$$

$$\eta_V = \frac{1}{2}(\chi_{+1} - \chi_{-1})\cos\gamma$$

and,

$$\rho_Q = \frac{1}{2}\left\{\widetilde{\chi}_0 - \frac{\widetilde{\chi}_{+1}+\widetilde{\chi}_{-1}}{2}\right\}\sin^2\gamma\cos 2\varphi$$

$$\rho_U = \frac{1}{2}\left\{\widetilde{\chi}_0 - \frac{\widetilde{\chi}_{+1}+\widetilde{\chi}_{-1}}{2}\right\}\sin^2\gamma\sin 2\varphi$$

$$\rho_V = \frac{1}{2}(\widetilde{\chi}_{+1} - \widetilde{\chi}_{-1})\cos\gamma$$

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Absorption & dispersion: χ & $\widetilde{\chi}$

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- $\chi_\alpha \Rightarrow \eta \Rightarrow$ absoprtion profiles $\Rightarrow$ related to the conductivity matrix $\hat{\sigma} \Rightarrow$ Voigt function
- $\widetilde{\chi}_\alpha \Rightarrow \rho \Rightarrow$ dispersion profiles $\Rightarrow$ related to the displacement matrix $\hat{\mathcal{E}} \Rightarrow$ Faraday function
- $\alpha = [-1, 0, +1]$ are the so-called principal axis of the medium $\Rightarrow$ basis $[\mathbf{e}_\alpha]$ diagonalizes $\hat{\sigma}$ and $\hat{\mathcal{E}} \Rightarrow$ related to the classical Zeeman effect.

The principal axes of a conducting medium under $\mathbf{B}_{\rm ext}$

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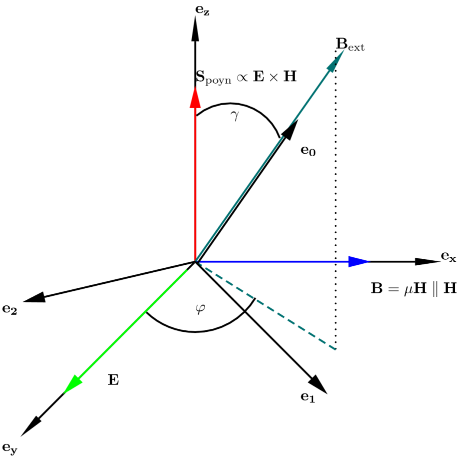
We define a base $[\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2]$ where $\mathbf{e}_0 \parallel \mathbf{B}_{\rm ext}$



The principal axes of a conducting medium under $\mathbf{B}_{\rm ext}$

We further define:

$$\begin{aligned} \mathbf{e}_0 &= \mathbf{e}_0 \\ \mathbf{e}_{+1} &= \frac{1}{\sqrt{2}}(\mathbf{e}_1 + i\mathbf{e}_2) \\ \mathbf{e}_{-1} &= \frac{1}{\sqrt{2}}(\mathbf{e}_1 - i\mathbf{e}_2) \end{aligned}$$

It is easy to demonstrate that:

$$\mathbf{e}_0 \times \mathbf{e}_0 = 0$$

$$\mathbf{e}_{+1} \times \mathbf{e}_0 = i\mathbf{e}_{+1}$$

$$\mathbf{e}_{-1} \times \mathbf{e}_0 = -i\mathbf{e}_{-1}$$

In compact form:

$$\boxed{\mathbf{e}_\alpha \times \mathbf{e}_0 = i\alpha\mathbf{e}_\alpha} \tag{15}$$

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Finding χ_α and $\widetilde{\chi}_\alpha$

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Remember this equation ? Equation of motion of an electron with $\mathbf{B}_{\rm ext}$

$$\mathbf{F} = m\ddot{\mathbf{r}} = -k\mathbf{r} - e\mathbf{E}_{\rm ext} - m\Gamma\dot{\mathbf{r}} - \frac{e}{c}\dot{\mathbf{r}} \times \mathbf{B}_{\rm ext}$$

- We proceed exactly as before but we work on the reference frame of the principal axes.
- The α-component of this solution is:

$$\begin{aligned} r_\alpha &= r_{\alpha 0}{\rm e}^{-iwt} \\ \dot{r}_\alpha &= -iwr_\alpha \\ \ddot{r}_\alpha &= -w^2 r_\alpha \end{aligned}$$

- Insering this solution into Eq.(14), and using Eq.(15), we obtain:

$$-mw^2 r_\alpha = -kr_\alpha + imw\Gamma r_\alpha - eE_\alpha + \frac{ew\alpha B}{c} r_\alpha$$

Finding χ_α and $\widetilde{\chi}_\alpha$

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- Solving for r_α we obtain:

$$r_\alpha = \frac{-eE_\alpha}{k - mw^2 - imw\Gamma - \frac{e}{c}w\alpha B} \tag{16}$$

- Note that the ratio E_α / r_α depends on $\alpha \Rightarrow$ $E_0/r_0 \neq E_{+1}/r_{+1} \neq E_{-1}/r_{-1}$
- $\mathbf{r} \not\parallel \mathbf{E} \Rightarrow \hat{\varepsilon}$ is a matrix.
- But, $r_\alpha \propto E_\alpha$, therefore $\hat{\varepsilon}$ is a diagonal matrix. $\Rightarrow$ The frame of the principal axes diagonalizes $\hat{\varepsilon}$
- We can now obtain ε_α

$$D_\alpha = E_\alpha - 4\pi N e r_\alpha = \{\text{Eq.}(16)\} =$$

$$= \left\{1 + \frac{4\pi N e^2}{k - mw^2 - imw\Gamma - \frac{e}{c}w\alpha B}\right\} E_\alpha = \varepsilon_\alpha E_\alpha$$

Finding χ_α and $\widetilde{\chi}_\alpha$

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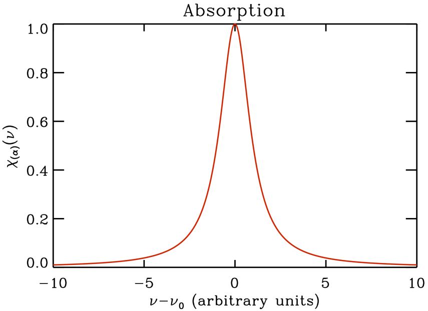
$$\begin{aligned} \mathrm{Re}(\varepsilon_\alpha) &= \chi_\alpha \Rightarrow \text{absorption profile} \\ \mathrm{Im}(\varepsilon_\alpha) &= \widetilde{\chi}_\alpha \Rightarrow \text{dispersion profile} \end{aligned}$$

$$\begin{aligned} \chi_\alpha &= \frac{Ne^2}{4\pi m} \frac{\Gamma}{(w_p - w + w_\alpha)^2 + \Gamma^2} \\ \widetilde{\chi}_\alpha &= 1 + \frac{Ne^2}{2\pi m} \frac{w_p - w + w_\alpha}{(w_p - w + w_\alpha)^2 + \Gamma^2} \\ w_p &= \sqrt{k/m} \\ w_\alpha &= \frac{\alpha e B_{\rm ext}}{mc} \end{aligned} \tag{17}$$

Interpretation of χ_α



- Single Lorentzian ⇒ Voigt (if collisions and thermal motions are included)
- It removes energy from the electric field of the passing EM wave through Joule dissipation

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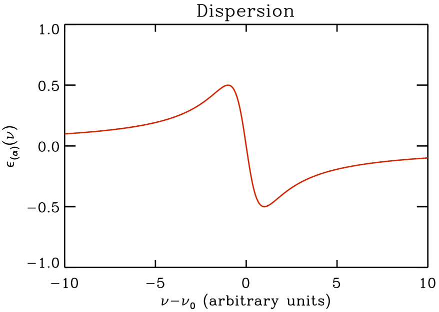
Interpretation of $\widetilde{\chi}_\alpha$



- Double Lorentzian $\Rightarrow$ double Voigt (if collisions and thermal motions are included)
- It does not remove energy from the electric field of the passing EM wave
- It introduces lags between E_x and E_y $\Rightarrow$ it modifies the polarization properties of the passing EM wave

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Interpretation of w_p (plasma frequency)

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"Classical" electron transitions in an atom

- In the absence of damping ($\Gamma = 0$) and no external magnetic field ($B_{\rm ext} = 0$), Eq(16) becomes:

$$r_\alpha = \frac{-eE_\alpha}{k - mw^2}$$

- And if $w = w_p = \sqrt{k/m}$, then $r_\alpha \rightarrow \infty$
- This means the electron jumps away from its equilibrium position $\mathbf{r}_0$
- w_p can be interpreted as $w_0 = 2\pi\nu_0$ with $\nu_0 = \hbar^{-1}(E_{\rm upp} - E_{\rm low})$
- This can be seen as the electron moving into another energy level within the atom (bound-bound transition) or away from it (ionization or bound-free transition)

Classical Zeeman effect

- In the presence of an external magnetic field $\mathbf{B}_{\rm ext}$ the absorption χ_α and dispersion $\widetilde{\chi}_\alpha$ profiles split in three.
  - $\alpha = 0$ remains unshifted: π-transition
  - $\alpha = 1$ shifts by an amount $eB_{\rm ext}/mc$ towards a larger w (lower λ): σ_b-transition
  - $\alpha = -1$ shifts by an amount $-eB_{\rm ext}/mc$ towards a lower w (larger λ): σ_r-transition
- This is the so-called normal (classical) Zeeman effect $\Rightarrow$ splitting of energy levels in the presence of $\mathbf{B}_{\rm ext}$.

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Quantum treatment of the Zeeman effect

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In the presence of $B_{\rm ext}$ all energy levels involved in a transition split according to:

$$\begin{aligned} E_{\rm new} &= E_{\rm old} - \frac{eB_{\rm ext}\hbar}{2mc} m_j g(j,l,s) \\ g(j,l,s) &= \frac{3j(j+1) - l(l+1) + s(s+1)}{2j(j+1)} \end{aligned} \quad (18)$$

- Energy splitting depents on the quantum numbers s, l, j, m_j, with $j = s + l$ and $m_j = -j, .., ..., +j$.
- $g(j,l,s)$ is the Landé factor
- Not all transitions between all new energy levels are allowed $\Rightarrow$ transition rules
- Electric dipole transition rules:

$$\Delta j = 0, \pm 1 \;\; ; \;\; \Delta m_j = 0, \pm 1 \quad (19)$$

The normal Zeeman pattern

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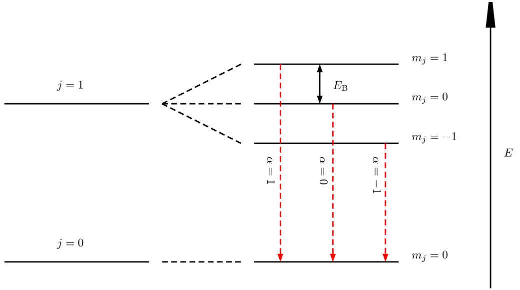
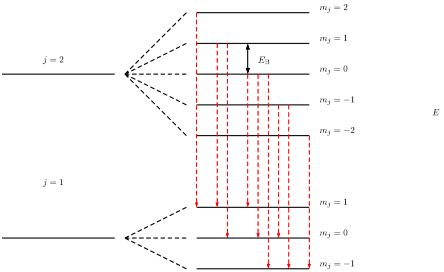
If transition $j = 0 \rightarrow j = 1$ we have a normal Zeeman pattern



- The normal Zeeman pattern can be treated using the classical theory
- Three new spectral lines appear: $\Delta m_j = -1, 0, 1$ correspond to $\alpha = -1, 0, 1$, respectively.

The anomalous Zeeman pattern

If transition is not of the type $j = 0 \rightarrow j = 1$ we have an anomalous Zeeman pattern



- $2\min(j_u, j_l) + 1$ transitions with $\Delta m_j = 0 \Rightarrow \alpha = 0 \Rightarrow \chi_0$ and $\widetilde{\chi}_0$
- $j_u + j_l$ transitions with $\Delta m_j = 1 \Rightarrow \alpha = +1 \Rightarrow \chi_{+1}$ and $\widetilde{\chi}_{+1}$
- $j_u + j_l$ transitions with $\Delta m_j = -1 \Rightarrow \alpha = -1 \Rightarrow \chi_{-1}$ and $\widetilde{\chi}_{-1}$

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The effective Zeeman triplet

- The energy of transitions with same Δm_j is much closer than the energy of transitions with different Δm_j.
- In many case the anomalous Zeeman pattern can be treated as a normal Zeeman pattern with just three spectral lines $\Rightarrow$ effective Zeeman triplet
- If the magnetic field is large or thermal width of the spectral line low $\Rightarrow$ we cannot use the effective Zeeman triplet

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A simple solution for the radiative transfer equation

Strong/saturated spectral lines

- A spectral line is saturated if:
  - the element's abundance is very large and/or
  - the transition probability between two electronic energy levels is very large
- In this case, it can be demonstrated that:

$$I \sim \eta_I \;\; ;Q \sim \eta_Q \;\; ;U \sim \eta_U \;\; ;V \sim \eta_V$$

- Let us look at several examples:

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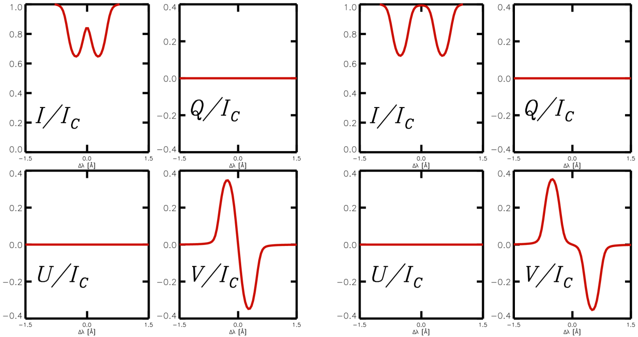
Vertical magnetic field $\gamma = 0 \Rightarrow$ Eq.(11)



$$\eta_I = \frac{1}{2}\{\chi_{+1} + \chi_{-1}\}$$

$$\eta_V = \frac{1}{2}(\chi_{+1} - \chi_{-1})$$

$$\eta_Q = \eta_U = 0$$

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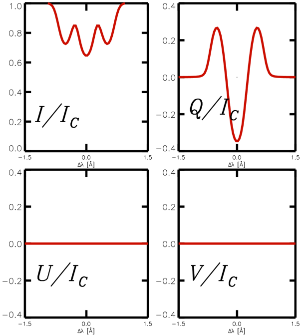
Horizontal magnetic field $\gamma = 90$; $\varphi = 0 \Rightarrow$ Eq.(11)



$$\eta_I = \frac{1}{2}\left\{\chi_0 + \frac{\chi_{+1} + \chi_{-1}}{2}\right\}$$

$$\eta_Q = \frac{1}{2}\left\{\chi_0 - \frac{\chi_{+1} + \chi_{-1}}{2}\right\}$$

$$\eta_U = \eta_V = 0$$

Horizontal magnetic field $\gamma = 90$; $\varphi = 45 \Rightarrow$ Eq.(11)

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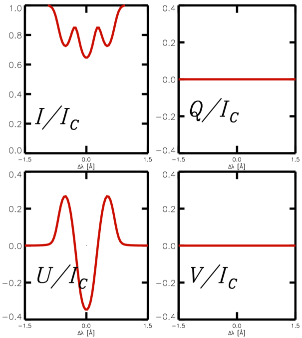

$$\eta_I = \frac{1}{2}\left\{\chi_0 + \frac{\chi_{+1} + \chi_{-1}}{2}\right\}$$

$$\eta_U = \frac{1}{2}\left\{\chi_0 - \frac{\chi_{+1} + \chi_{-1}}{2}\right\}$$

$$\eta_Q = \eta_V = 0$$

Numerical solvers in optical depth τ_c

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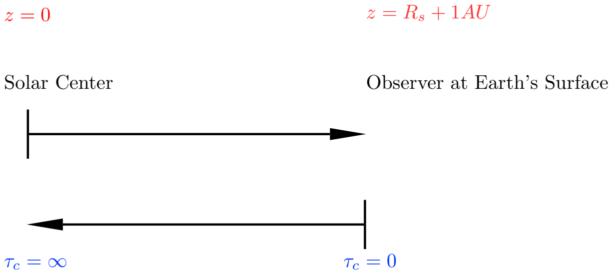
Most codes that solve the RTE do it in the continuum optical-depth scale τ_c instead of the geometrical z scale. Define τ_c as:

$$d\tau_c = -\rho \mathcal{K}_c dz \tag{20}$$

Where ρ is the density and $\mathcal{K}_c$ is the continuum opacity: bound-free (i.e. ionization) and free-free (i.e. scattering) transtions $\Rightarrow$ no spectral lines. Usually the continuum is evaluated at a wavelength of 500 nm: $\mathcal{K}_c \Rightarrow \mathcal{K}_5$ and $\tau_c \Rightarrow \tau_5$



Numerical solvers in optical depth τ_c

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Typical solvers are:

- Runge-Kutta Wittmann (1974), Landi Delg'Innocenti (1974): 4th-order accuarte
- DELO (Rees et al. 1989) linear, parabolic (Murphy 1990), cubic, Bézier (De la Cruz Rodriguez & Piskunov 2013): 2nd-3rd order accurate
- Hermitian (Bellot Rubio et al. 1998): 4th-order accurate
- (W)ENO methods (Jannet et al. 2019): 4th-order accurate

We will be leaning the **SIR** code (Ruiz Cobo & Del Toro Iniesta 1992) $\Rightarrow$ 4th-order Hermitian algorithm.

Problems with solvers in τ_c

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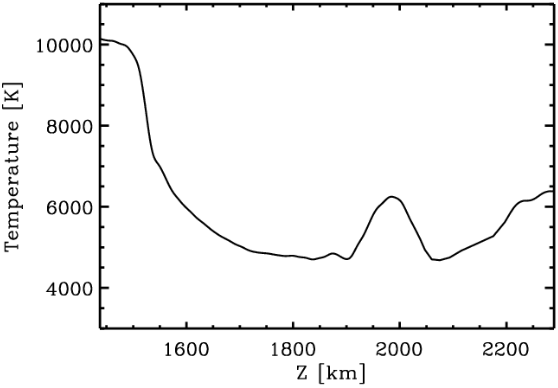
MHD simulations provide the physical parameters as a function of (x, y, z)

- Conversion from z to τ_c according to Eq.(20) is needed.
- Often physical parameters change rapidly between adjacent optical depth points after conversion.
- This makes integration schemes (even high-order ones) blow up.
- Need to reinterpolate atmospheres in z to a very fine grid before conversion to τ_c $\Rightarrow$ speed loss

Problems with solvers in τ_c



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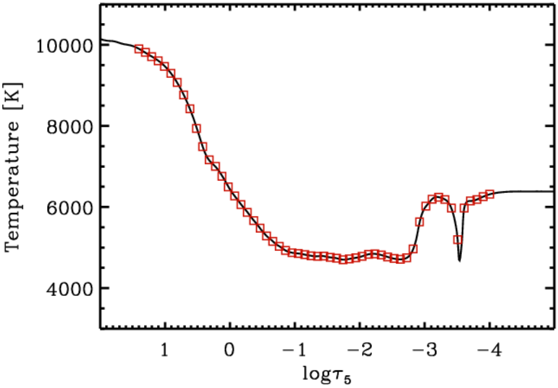
Problems with solvers in τ_c



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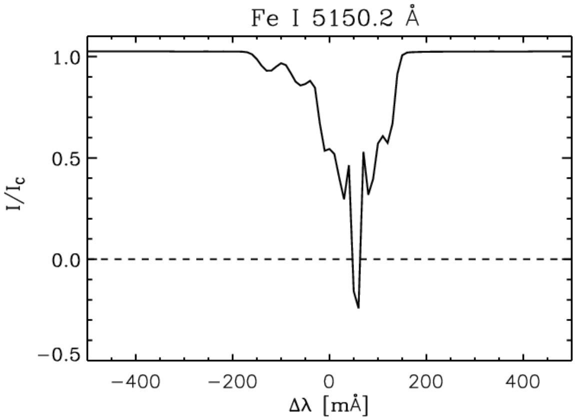
Problems with solvers in τ_c



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