1st SolarNet school: formation of specral lines in solar/stellar atmospheres

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Radiative transfer equation for intensity

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Definition

$$I(\nu) = \text{specific intensity}$$
$$I(\nu) = \frac{\text{energy}}{\text{time} * \text{area} * \text{frequency} * \text{solid angle}}$$

Phenomenological RTE derivation

The RTE tells us how the energy contained in the radiation field changes along the ray path:

$$\frac{\mathrm{d}I(\nu, s)}{\mathrm{d}s} = \text{Sources} - \text{Sinks}$$
$$s = \text{coordinate along the ray path}$$

Sources and sinks

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Sources

$$\text{Sources} = j(\nu, s) = \text{specific emissivity}$$

$$j(\nu, s) = \frac{\text{energy emitted by the medium}}{\text{time} * \text{area} * \text{frequency} * \text{solid angle} * \text{lengh}}$$

Sinks

$$\text{Sources} = \eta(\nu, s) = \text{specific absorption}$$

$$\eta(\nu, s) = \frac{\text{energy absorved by the medium}}{\text{time} * \text{area} * \text{frequency} * \text{solid angle} * \text{lengh}}$$

More on sinks

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Sinks

If there is no light then it cannot be absorbed:

$$\eta(\nu, s) \propto I(\nu, s)$$
$$\eta(\nu, s) = \mathcal{K}(\nu, s) I(\nu, s)$$

Absorption coefficient

$$\mathcal{K}(\nu, s) = \frac{1}{\color{red}{\text{length}}}$$

Optical depth

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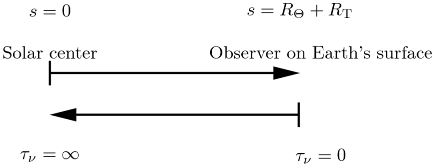
RTE in terms of s:

$$\boxed{\frac{\mathrm{d}I(\nu,s)}{\mathrm{d}s} = j(\nu,s) - \mathcal{K}(\nu,s)I(\nu,s)} \qquad (1)$$

In red we indicate the parameters that depend on the properties of the medium (density, pressure, temperature, chemical composition, magnetic field, etc).

We define optical depth $\tau(\nu, s)$ as:

$$\mathrm{d}\tau(\nu,s) = -\mathcal{K}(\nu,s)\mathrm{d}s \quad (\tau \text{ dimensionless}) \qquad (2)$$



Optical depth

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Substituting Eq. 2 into Eq. 1 we obtain:

$$\frac{1}{\mathcal{K}(\nu,s)}\frac{\mathrm{d}I(\nu,s)}{\mathrm{d}s} = \frac{j(\nu,s)}{\mathcal{K}(\nu,s)} - I(\nu,s)$$

$$-\frac{\mathrm{d}I(\nu,s)}{\mathrm{d}\tau(\nu,s)} = \underbrace{\frac{j(\nu,s)}{\mathcal{K}(\nu,s)}}_{S(\nu,s)} - I(\nu,s)$$

$S(\nu,s)$ is the so-called *Source function* and corresponds to the ratio between the specify emissivity $j(\nu,s)$ and absorption $\mathcal{K}(\nu,s)$.

RTE in terms of τ:

$$\boxed{\frac{\mathrm{d}I(\tau)}{\mathrm{d}\tau} = I(\tau) - S(\tau)} \qquad (3)$$

Formal solution of the RTE

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The RTE is a first order inhomogeneous differential equation:

In principle it can be solved using an integrating factor.

$$e^{-\tau}\frac{dI(\tau)}{d\tau} = e^{-\tau}\{I(\tau)-S(\tau)\}$$

$\rightarrow$ re-agrouping terms

$$\frac{d}{d\tau}\left\{I(\tau)e^{-\tau}\right\} = -S(\tau)e^{-\tau}$$

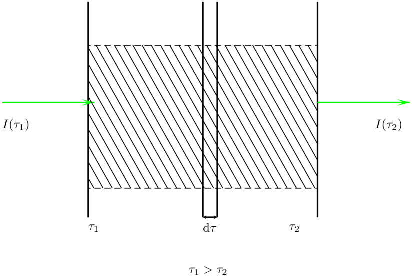
$\rightarrow$ direct integration between τ_1 and τ_2

$$\int_{\tau_1}^{\tau_2} d\{I(\tau)e^{-\tau}\} = -\int_{\tau_1}^{\tau_2} S(\tau)e^{-\tau}d\tau$$

Formal solution of the RTE

Easily leads to:

$$\boxed{I(\tau_2) = I(\tau_1)\mathrm{e}^{\tau_2-\tau_1} - \int_{\tau_1}^{\tau_2} S(\tau)\mathrm{e}^{\tau_2-\tau}\mathrm{d}\tau} \qquad (4)$$



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Survival probability

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If we ignore sources, $S(\tau) = 0$, in Eq. 4, the solution to the RTE becomes:

$$\begin{aligned} I(\tau_2) &= I(\tau_1) \mathrm{e}^{\tau_2 - \tau_1} \quad \text{with } \tau_1 > \tau_2 \\ \frac{I(\tau_2)}{I(\tau_1)} &= \mathrm{e}^{\tau_2 - \tau_1} \end{aligned}$$

On the other hand:

$$\begin{aligned} \frac{I(\tau_2)}{I(\tau_1)} &= \frac{\text{Energy per unit ... at } \tau_2}{\text{Energy per unit ... at } \tau_1} \\ &= \frac{\text{Number of photons per unit ... at } \tau_2 * h\nu}{\text{Number of photons per unit ... at } \tau_1 * h\nu} \end{aligned}$$

Survival probability

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Finally we arrive to:

$$\frac{\text{Number of photons per unit ... at } \tau_2}{\text{Number of photons per unit ... at } \tau_1} = e^{\tau_2-\tau_1}$$

For instance:

If $\tau_1 - \tau_2 = 1$ (remember $\tau_1 > \tau_2$), then:

$$\frac{\text{Number of photons per unit ... at } \tau_2}{\text{Number of photons per unit ... at } \tau_1} \approx 0.368$$

About 37 % of the photons at any given frecuency ν survive after travelling one unit of optical depth (if no there are no sources of photons in between).

Concept of $\overline{\Delta\tau} = 1$

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Question: what is the mean optical depth travelled by the photons we observe at some frequency ν ?

We make $\tau_1 = \infty$ (Sun's or star's interior) and $\tau_2 = 0$ (observer's location):

$$\overline{\Delta\tau} = \frac{\int_0^\infty \Delta\tau p(\Delta\tau) \mathrm{d}(\Delta\tau)}{\int_0^\infty p(\Delta\tau) \mathrm{d}(\Delta\tau)} = \left\{ p(\Delta\tau) \text{ is the survival probability} \right\} =$$

$$= \frac{\int_0^\infty \Delta\tau \mathrm{e}^{-\Delta\tau} \mathrm{d}(\Delta\tau)}{\int_0^\infty \mathrm{e}^{-\Delta\tau} \mathrm{d}(\Delta\tau)} = \left\{ \text{integration by parts} \right\} = 1$$

Answer: the mean optical depth tavelled by the observed photons is $\overline{\Delta\tau}$=1 (in the absence of sources). This stands for all frequencies.

What geometrical depth is $\overline{\Delta\tau} = 1$?

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Remember relation between geometrial s and optical depth τ (Eq. 2)

$$\mathrm{d}\tau(\nu, s) = -\mathcal{K}(\nu, s)\mathrm{d}s$$

$\rightarrow$ integrate between s_1 and s_2

$$\int_{s_1}^{s_2} \mathrm{d}\tau(\nu, s) = -\int_{s_1}^{s_2} \mathcal{K}(\nu, s)\mathrm{d}s$$

$\rightarrow$ take mean absorption coefficient on the rhs

$$\tau(\nu, s_2) - \tau(\nu, s_1) \approx -\widetilde{\mathcal{K}}(\nu)\Delta s$$

$\rightarrow$ take mean of the photon ensemble

$$\overline{\tau(\nu, s_2) - \tau(\nu, s_1)} \approx -\overline{\widetilde{\mathcal{K}}}(\nu)\overline{\Delta s} = 1 \text{ !!!!}$$

$\rightarrow$ =1 stands for all possible frequencies

What geometrical depth is $\overline{\Delta\tau} = 1$?

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Finally we arrive at:

$$\overline{\Delta s} \approx -\frac{1}{\overline{\overline{\mathcal{K}}}(\nu)} \rightarrow \text{frequency dependent} \qquad (5)$$

Take ratio at two dfifferent frequencies:

$$\frac{\overline{\Delta s_1}}{\overline{\Delta s_2}} \approx \frac{\overline{\overline{\mathcal{K}}}(\nu_2)}{\overline{\overline{\mathcal{K}}}(\nu_1)}$$

- If absorption at $\nu_2 \gg \nu_1$ then $\overline{\Delta s_1} \gg \overline{\Delta s_2}$. Therefore, photons with frequency ν_1 have travelled much more before being observed: they come from further (i.e. deeper in the sun/star) than those with ν_2.
- If absorption at $\nu_2 \ll \nu_1$ then $\overline{\Delta s_1} \ll \overline{\Delta s_2}$. Therefore, photons with frequency ν_2 have travelled much less before being observed: they come from closer (i.e not so deep in the sun/star) than those with ν_1.

Absorption coefficient $\mathcal{K}(\nu)$: adding spectral lines

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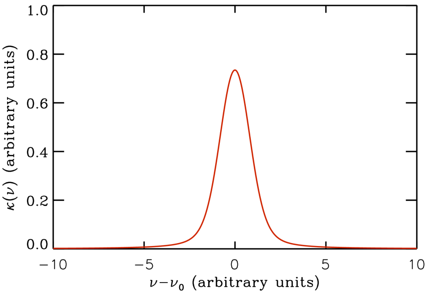
- $\mathcal{K}(\nu) \propto$ Probability of absorption of a photon by an atom
- Probability is not a δ-Dirac located at $\nu_0 = \hbar^{-1}(E_{\rm upp} - E_{\rm low})$
  1. Energy levels have an intrinsic width due to the uncertainty principle: Lorentzian profile in ν
  2. Atom's thermal motion: Gaussian profile in ν
  3. Collisions between atoms: Lorentzian profile in ν
- Convolution of Lorentzian and Gaussian: Voigt profile in ν



Absorption coefficient $\mathcal{K}(\nu)$

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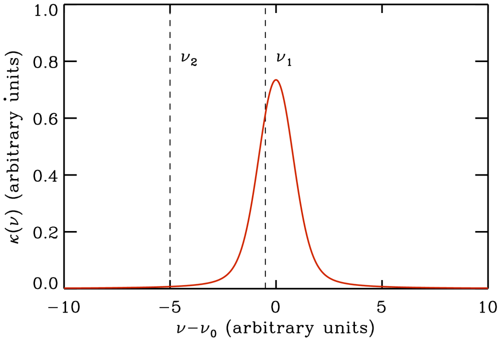

1. $\nu_2 \gg \nu_0 \rightarrow \mathcal{K}(\nu_1) \ll 1 \rightarrow$ observe photons coming from deep into the solar/stellar surface
2. $\nu_1 \approx \nu_0 \rightarrow \mathcal{K}(\nu_0) \approx 1 \rightarrow$ observe photons closer to us

How does the intensity $I(\nu)$ look like ?

Absorption lines: Photosphere

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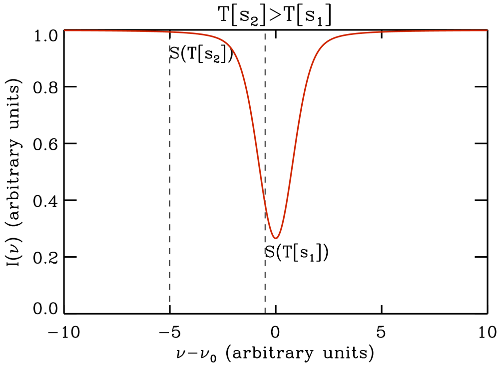
Temperature decreases outwards

$$\frac{\mathrm{d}T(s)}{\mathrm{d}s} < 0$$

$$S(T[s],\nu) = \frac{2h\nu^3}{c^2}\frac{1}{\mathrm{e}^{\frac{h\nu}{KT[s]}}-1} \quad \text{(Source function Eq. 3)}$$



Emissions lines: Corona

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Temperature increases outwards

$$\frac{\mathrm{d}T(s)}{\mathrm{d}s} > 0$$

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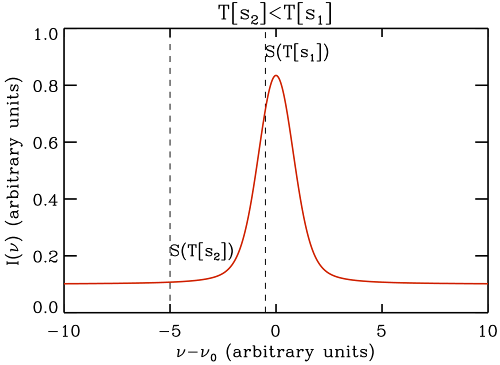
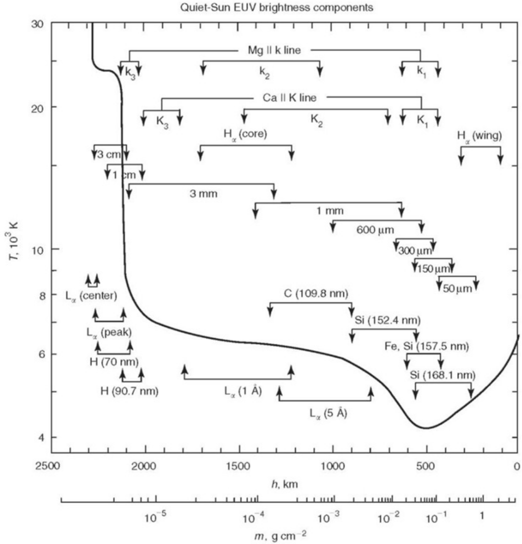


The solar atmosphere: absorption/emission



Back to core emission

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A very easy situation: blackbody radiation

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constant $T[s]$

$$T[s] = T_0 \quad \forall s$$

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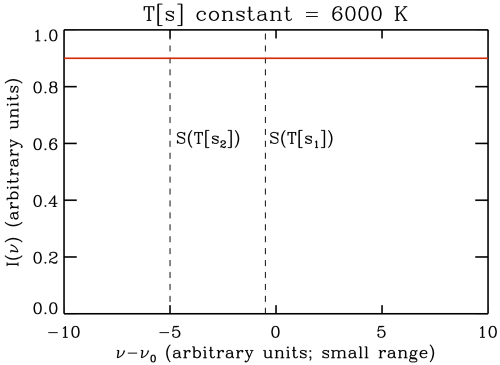

A very easy situation: black body radiation

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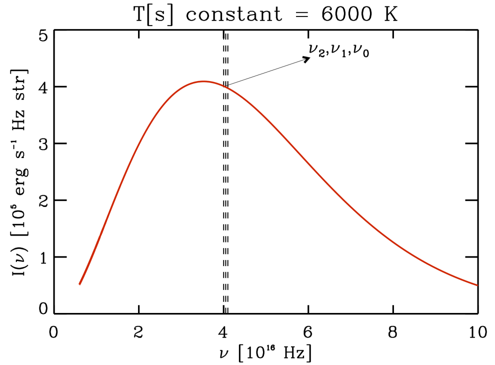

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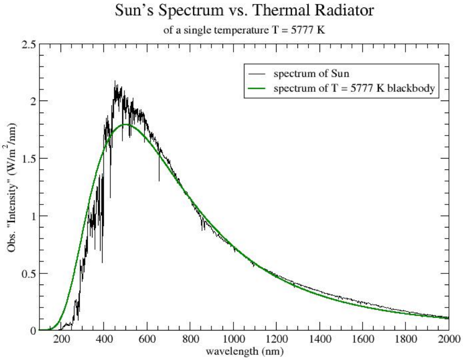

Absorption with core emission

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T decreases and then increases

$s_2 > s_1 > s_0$ but $T[s_2] > T[s_0] > T[s_1]$

$$S(T[s],\nu) = \frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{KT_0}}-1} \quad \text{(Source function Eq. 3)}$$